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Power Series for Solutions of the $3\mathcal{D}$-Navier-Stokes System on R^3

Yakov Sinai[1]

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In this paper we study the Fourier transform of the $3\mathcal{D}$-Navier-Stokes System without external forcing on the whole space R^3. The properties of solutions depend very much on the space in which the system is considered. In this paper we deal with the space $\Phi(\alpha,\alpha)$ of functions $v(k) = \frac{c(k)}{|k|^\alpha}$ where $\alpha = 2+\epsilon$, $\epsilon > 0$ and $c(k)$ is bounded, $\sup_{k\in R^3 \setminus 0} |c(k)| < \infty$. We construct the power series which converges for small t and gives solutions of the system for bounded intervals of time. These solutions can be estimated at infinity (in k-space) by $\exp\{-\text{const}\sqrt{t}|k|\}$.

KEY WORDS: Navier-Stokes System; Fourier transform; power series.

1. THE SPACES $\Phi(\alpha,\alpha)$ AND THE RULING PARAMETER FOR THE NAVIER-STOKES SYSTEM IN $\Phi(\alpha,\alpha)$

Consider $3\mathcal{D}$-Navier-Stokes System (NSS) on R^3 for incompressible free fluids. After Fourier transform and an elementary transformation it becomes a non-linear non-local equation for an unknown function $v(k,t)$ with values in R^3 having the form

$$\begin{aligned} v(k,t) &= e^{-t|k|^2} v(k,0) + i\int_0^t e^{-|k|^2(t-s)}\, ds \\ &\quad \cdot \int_{R^3} \langle k, v(k-k',s)\rangle\, P_k\, v(k',s)\, dk' . \end{aligned} \tag{1}$$

[1]Department of Mathematics, Princeton University, Princeton, NJ, USA and Landau Institute of Theoretical Physics, Moscow, Russia; e-mail: sinai@math.princeton.edu

The function $v(k,t)$ must satisfy the condition $v(k,t) \perp k$ for any $k \in R^3, k \neq 0$ and $t \geqslant 0$, P_k is the orthogonal projection to the subspace orthogonal to k, the viscosity is taken to be one, i.e., $\nu = 1$. Classical solutions of (1) on $[0, t_0]$ are functions $v(k,t), 0 \leqslant t \leqslant t_0$, for which all integrals in (1) converge absolutely and (1) becomes the identity.

There are several reasons by which it is natural to consider (1) in the spaces of functions having singularities near $k=0$ or $k=\infty$. In this paper, we restrict ourselves to the spaces of functions $v(k) = \frac{c(k)}{|k|^\alpha}$ where $\alpha = 2+\epsilon, \epsilon > 0$ and sufficiently small, $c(k)$ is continuous everywhere outside $k=0$ and uniformly bounded, i.e., $\sup_{k \in R^3 \setminus 0} |c(k)| = \| c \| < \infty$ (see refs. 1 and 7). If the solution of (1) belongs to $\Phi(\alpha, \alpha)$ then $v(k,t) = \frac{c(k,t)}{|k|^\alpha}, 0 \leqslant t \leqslant t_0, c(k,t) \perp k$ for any $k \in R^3 \setminus 0, t \geqslant 0$, and $c(k,t)$ satisfies the equation which is equivalent to (1):

$$c(k,t) = e^{-|k|^2 t} c(k,0) + i|k|^\alpha \int_0^t e^{-|k|^2(t-s)} ds \cdot \int_{R^3} \frac{\langle k,\, c(k-k',\, s) \rangle \; P_k\, c(k',s)\, dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}. \tag{2}$$

It is easy to check that for typical $c \in \Phi(\alpha, \alpha)$ the initial condition has infinite energy and enstrophy.

Assume that $\|c(k,0)\| = 1$ and take one-parameter family of initial conditions $c_A(k,0) = A\, c(k,0)$ where A is a parameter taking positive values. In ref. 1 the local existence theorem for solution of (2) was proven. Below we outline this proof and show that if $\lambda = A \cdot t^{\epsilon/2} \leqslant \lambda_0$ where λ_0 is an absolute constant which may depend on α, then there exists the unique solution of (2) on the corresponding time interval.

Usual arguments are based on the classical iteration scheme. Put $c_A^{(0)}(k,t) = A\, e^{-|k|^2 t} c(k,0)$ and

$$c_A^{(n)}(k,t) = c_A^{(0)}(k,t) + i|k|^\alpha \int_0^t e^{-|k|^2(t-s)} ds \cdot \int_{R^3} \frac{\langle k, c_A^{(n-1)}(k-k',s) \rangle \, P_k\, c_A^{(n-1)}(k',s)\, dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}. \tag{3}$$

The first step in the proof of the convergence of the iterations $c_A^{(n)}(k,t)$ as $n \to \infty$ is to show that all $c_A^{(n)}(k,t)$ remain close to $c_A^{(0)}(k,t)$ in the sense of the norm in $\Phi(\alpha, \alpha)$. If $c_A^{(n)} = \sup_{\substack{0 \leqslant s \leqslant t \\ k \in R^3 \setminus 0}} |c_A^{(n)}(k,s)|$ then we would like to show that $c_A^{(n)} \leqslant 2c_A^{(0)} = 2A$ for all n. By induction and with the use of (3) we can write

$$c_A^{(n)} \leqslant c_A^{(0)} + \sup_{\substack{k\in R^3\setminus 0\\ 0\leqslant s\leqslant t}} |k|^{\alpha-1} \cdot (1 - e^{-|k|^2 t}) \cdot (c_A^{(n-1)})^2 \cdot \int \frac{dk'}{|k-k'|^{\alpha} \cdot |k'|^{\alpha}}. \tag{4}$$

The last integral satisfies the inequality

$$\int_{R^3} \frac{dk'}{|k-k'|^{\alpha} \cdot |k'|^{\alpha}} \leqslant \frac{B_1}{|k|^{2\alpha-3}}. \tag{5}$$

Here and below the letter B with indices is used for various absolute constants which appear during the proofs. These constants may depend on α.

Now, we have to show that

$$\sup_k |k|^{2-\alpha}(1-e^{-|k|^2 t}) \cdot (c_A^{(n-1)})^2 \cdot B_1 \leqslant c_A^{(0)}.$$

By induction $c_A^{(n-1)} \leqslant 2c_A^{(0)}$. Therefore we have to show that

$$\sup_k |k|^{2-\alpha}(1-e^{-|k|^2 t}) \cdot 4 \cdot B_1 \cdot c_A^{(0)} B_1 \leqslant 1\,.$$

Consider two cases.

1. $|k|^2 \leqslant \frac{1}{t}$. Then

$$|k|^{2-\alpha}(1-e^{-|k|^2 t}) \leqslant |k|^{4-\alpha} \cdot t \leqslant t^{-\frac{4-\alpha}{2}+1} = t^{\frac{\epsilon}{2}}\,.$$

2. $|k|^2 \geqslant \frac{1}{t}$. Then

$$|k|^{2-\alpha} \cdot (1-e^{-|k|^2 t}) \leqslant |k|^{2-\alpha} \leqslant t^{\frac{\alpha-2}{2}} = t^{\frac{\epsilon}{2}}\,.$$

Thus we can write

$$c_A^{(n)} \leqslant A \cdot + 4A^2 \cdot (c^{(0)})^2 \cdot t^{\frac{\epsilon}{2}} B_1 = A(1 + 4A \cdot t^{\frac{\epsilon}{2}} B_1) = A(1 + 4B_1\lambda)\,.$$

We used the fact that $c^{(0)}=c_1^{(0)}\leqslant 1$. If $\lambda<\frac{1}{4B_1}$ then $c_A^{(n)}\leqslant 2c_A^{(0)}$. This argument shows how the parameter λ arises.

The next step in the proof of the existence of solutions is to show that the iterations $c_A^{(n)}$ converge to a limit. We have from (3)

$$c_A^{(n)}(k,t)-c_A^{(n-1)}(k,t)=i\,|k|^{\alpha}\cdot\int_0^t e^{-|k|^2(t-s)}\,ds$$
$$\cdot\left[\int_{R^3}\frac{\langle k,c_A^{(n-1)}(k-k',s)-c_A^{(n-2)}(k-k',s)\rangle\,P_k\,c_A^{(n-1)}(k',s)\,dk'}{|k-k'|^{\alpha}\cdot|k'|^{\alpha}}\right.$$
$$\left.+\int_{R^3}\frac{\langle k,\,c_A^{(n-2)}(k-k',s)\rangle\,P_k\,(c_A^{(n-1)}(k',s)-c_A^{(n-2)}(k',s))\,dk'}{|k-k'|^{\alpha}\cdot|k'|^{\alpha}}\right]$$

and

$$|c_A^{(n)}(k,t)-c_A^{(n-1)}(k,t)|\leqslant 4A\cdot\|\,c_A^{(n-1)}-c_A^{(n-2)}\|$$
$$\cdot|k|^{\alpha-1}\,(1-e^{-|k|^2t})\int_{R^3}\frac{dk'}{|k-k'|^{\alpha}\cdot|k'|^{\alpha}}. \tag{6}$$

From (5) and (6)

$$\|c_A^{(n)}-c_A^{(n-1)}\|\leqslant\ 4A\,B_1\cdot\|c_A^{(n-1)}-c_A^{(n-2)}\|$$
$$\cdot\ \sup_{k\in R^3\setminus 0}\ |k|^{\alpha-1}\,(1-e^{-|k|^2t})\cdot\frac{1}{|k|^{2\alpha-3}}.$$

The same arguments as before give that

$$\sup_{k\in R^3\setminus 0}\ |k|^{\alpha-1}\,(1-e^{-|k|^2t})\cdot\frac{1}{|k|^{2\alpha-3}}\leqslant B_2\cdot t^{\frac{\epsilon}{2}}\,.$$

Therefore, for some constant B_3

$$\|c_A^{(n)}-c_A^{(n-1)}\|\leqslant B_3\cdot\lambda\cdot\|\,c_A^{(n-1)}-c_A^{(n-2)}\,\|\,.$$

From the last inequality it follows that if λ is less than some absolute constant then the iteration scheme converges and gives the desired solution. Thus λ is really a ruling parameter in the current situation. The main purpose of this paper is to construct a general power series in λ which provides the solution of (2) with the given initial condition.

Write down the solution of (2) with the initial condition $A\cdot c(k,0)$, $\|c(k,0)\|\leqslant 1$, in the form

$$c_A(k,t) = A\left(c(k,0)e^{-t|k|^2} + \int_0^t e^{-(t-s)|k|^2}\sum_{p\geqslant 1}\lambda^p h_p(k,s)\,ds\right)$$
$$= A\left(c(k,0)e^{-t|k|^2} + \sum_{p\geqslant 1}A^p\int_0^t e^{-(t-s)|k|^2}s^{\frac{p\epsilon}{2}}h_p(k,s)\,ds\right) \quad (7)$$

where now $\lambda = A\cdot s^{\epsilon/2}$. Substituting this expression into (2) we get the system of recurrent relations for h_p. Below we give the explicit formulas for h_1, h_2 and then the general formula for $h_p, p\geqslant 3$. We have

$$A^2 s^{\frac{\epsilon}{2}} h_1(k,s) = iA^2|k|^\alpha \int_{R^3}\frac{\langle k, c(k-k',0)\rangle P_k c(k',0)e^{-s|k-k'|^2-s|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}, \quad (8)$$

$$A^3 s^\epsilon h_2(k,s) = iA^3\cdot|k|^\alpha$$
$$\cdot\left[\int_0^s s_1^{\frac{\epsilon}{2}}ds_1\int_{R^3}\frac{\langle k, h_1(k-k',s_1)\rangle P_k c(k',0)\cdot e^{-(s-s_1)|k-k'|^2-s|k'|^2}dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\right.$$
$$\left.+\int_0^s s_2^{\frac{\epsilon}{2}}ds_2\int_{R^3}\frac{\langle k, c(k-k',0)\rangle P_k h_1(k',s_2)\,e^{-s|k-k'|^2-(s-s_2)|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\right] \quad (9)$$

and

$$A^{p+1}s^{\frac{p\epsilon}{2}}h_p(k,s) = iA^{p+1}\cdot|k|^\alpha\cdot\left[\int_0^s s_1^{\frac{p-1}{2}\epsilon}\,ds_1\right.$$
$$\cdot\int_{R^3}\frac{\langle k, h_{p-1}(k-k',s_1)\rangle P_k c(k',0)e^{-(s-s_1)|k-k'|^2-s|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}$$
$$+\int_0^s s_2^{\frac{(p-1)\epsilon}{2}}ds_2\int_{R^3}\frac{\langle k, c(k-k',0)\rangle P_k h_{p-1}(k',s_2)e^{-s|k-k'|^2-(s-s_2)|k'|^2}dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}$$
$$+\sum_{\substack{p_1,p_2\geqslant 1\\ p_1+p_2=p-1}}\int_0^s s_1^{p_1\frac{\epsilon}{2}}\,ds_1\int_0^s s_2^{p_2\frac{\epsilon}{2}}\,ds_2$$
$$\left.\cdot\int_{R^3}\frac{\langle k, h_{p_1}(k-k',s_1)\rangle P_k h_{p_2}(k',s_2)\,e^{-(s-s_1)|k-k'|^2-(s-s_2)|k'|^2}dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\right]. \quad (10)$$

Use the following Ansatz: $h_p(k,s)=s^{\frac{\epsilon}{2}}|k|^\alpha g_p(k\sqrt{s},s)$ and in all integrals above make the change of variables: $s_1=s\cdot\tilde{s}_1, s_2=s\cdot\tilde{s}_2, k\sqrt{s}=\tilde{k}, k'\sqrt{s}=\tilde{k}'$. Thus $h_p(k,s)=s^{\epsilon/2}|k|^\alpha g_p(\tilde{k},s)$. Instead of (8)–(10) we shall get the system of recurrent relations for the functions $g_p(\tilde{k},s)$:

$$A^2 s^\epsilon |k|^\alpha \cdot g_1(\tilde{k},s)=iA^2\cdot|k|^\alpha\cdot s^\epsilon \int_{R^3} \frac{\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right) e^{-|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha\cdot|\tilde{k}'|^\alpha}$$

or

$$g_1(\tilde{k},s)=i\int_{R^3} \frac{\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right) e^{-|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha\cdot|\tilde{k}'|^\alpha}, \tag{11}$$

$$\begin{aligned} A^3 s^{\frac{3\epsilon}{2}}\cdot|k|^\alpha g_2(\tilde{k},s)=iA^3\cdot|k|^\alpha s^{\frac{3\epsilon}{2}} \Bigg[\int_0^1 \tilde{s}_1^\epsilon\, d\tilde{s}_1 \\ \cdot\int_{R^3} \frac{\langle \tilde{k}, g_1((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\tilde{s}_1)\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right) e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha} \\ +\int_0^1 \tilde{s}_2^\epsilon\, d\tilde{s}_2 \int_{R^3} \frac{\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle g_1(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2) e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}\Bigg] \end{aligned}$$

or

$$\begin{aligned} g_2(\tilde{k},s)=&\int_0^1 \tilde{s}_1^\epsilon\, d\tilde{s}_1 \int_{R^3} \frac{\langle \tilde{k}, g_1((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\cdot\tilde{s}_1)\rangle\cdot P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right) e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha} \\ &+\int_0^1 \tilde{s}_2^\epsilon\, d\tilde{s}_2 \int_{R^3} \frac{\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_{\tilde{k}} g_1\left(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\right) e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}. \end{aligned} \tag{12}$$

For general $g_p(\tilde{k},s)$, $p\geqslant 3$

$$A^{p+1}\cdot s^{\frac{p+1}{2}\cdot\epsilon}|k|^{\alpha}\cdot g_p(\tilde{k},s)=i\,A^{p+1}\cdot s^{\frac{p+1}{2}\cdot\epsilon}\cdot|k|^{\alpha}\Bigg[\int_0^1\tilde{s}_1^{\frac{p}{2}\epsilon}\,d\tilde{s}_1$$

$$\cdot\int_{R^3}\frac{\langle\tilde{k},g_{p-1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1},s\cdot\tilde{s}_1)\rangle P_{\tilde{k}}c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}'|^{\alpha}}$$

$$+\int_0^1\tilde{s}_2^{\frac{p\epsilon}{2}}\,d\tilde{s}_2\int_{R^3}\frac{\langle\tilde{k},c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_{\tilde{k}}g_{p-1}(\tilde{k}'\sqrt{\tilde{s}_2},s\tilde{s}_2)e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^{\alpha}}$$

$$+\sum_{\substack{p_1\geqslant 1,p_2\geqslant 1\\ p_1+p_2=p-1}}\int_0^1\tilde{s}_1^{\frac{(p_1+1)\epsilon}{2}}\,d\tilde{s}_1\int_0^1\tilde{s}_2^{\frac{(p_2+1)\epsilon}{2}}\,d\tilde{s}_2\int_{R^3}\langle\tilde{k},g_{p_1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1},s\tilde{s}_1))\rangle$$

$$\cdot P_{\tilde{k}}g_{p_2}(\tilde{k}'\sqrt{\tilde{s}_2},s\tilde{s}_2)\cdot e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}d\tilde{k}'$$

or

$$g_p(\tilde{k},s)=i\Bigg[\int_0^1\tilde{s}_1^{\frac{p\epsilon}{2}}\,d\tilde{s}_1$$

$$\cdot\int_{R^3}\frac{\langle\tilde{k},g_{p-1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1},s\tilde{s}_1)\rangle P_{\tilde{k}}c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}'|^{\alpha}}$$

$$+\int_0^1\tilde{s}_2^{\frac{p\epsilon}{2}}\,d\tilde{s}_2\int_{R^3}\frac{\langle\tilde{k},c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_{\tilde{k}}\,g_{p-1}(\tilde{k}'\sqrt{\tilde{s}_2},s\tilde{s}_2)e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^{\alpha}}$$

$$+\sum_{\substack{p_1,p_2\geqslant 1\\ p_1+p_2=p-1}}\int_0^1\tilde{s}_1^{\frac{(p_1+1)\epsilon}{2}}\,d\tilde{s}_1\int_0^1\tilde{s}_2^{\frac{(p_2+1)\epsilon}{2}}\,d\tilde{s}_2\int_{R^3}\langle\tilde{k},g_{p_1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1},s\cdot\tilde{s}_1)\rangle\,.$$

$$\cdot\,P_{\tilde{k}}g_{p_2}(\tilde{k}'\sqrt{\tilde{s}_2},s\tilde{s}_2)\cdot e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}\,d\tilde{k}'\Bigg].\tag{13}$$

These recurrent relations allow to express each $g_p(k\sqrt{s},s)$ through the initial conditions $c(k,0)$. It is easy to see that this expression will be the sum of not more than $b^p\,4p$-dimensional integrals containing products of $c(\cdot,0)$ with different values of the arguments where b is some constant. We shall discuss the related questions in another paper.

Write down the inequality:

$$|g_p(\tilde{k},s)|\leqslant C_p\,f(|\tilde{k}|)\,e^{-\frac{|k|^2 s}{p+1}}=C_p\,f(|\tilde{k}|)\,e^{-\frac{|\tilde{k}|^2}{p+1}}$$

where

$$f(x) = \begin{cases} x & 0 \leqslant x \leqslant 1, \\ x^{-1} & x \geqslant 1. \end{cases}$$

The main result of this paper is the following theorem.

Main Theorem. The numbers C_p can be chosen in such a way that

$$C_p = B \sum_{p_1+p_2=p-1} C_{p_1} \cdot C_{p_2} \cdot \frac{(p_1+1)(p_2+1)}{(p+1)}. \tag{14}$$

We prove the main theorem in Section 2. First we analyze $p = 1, 2, 3$ and then the general case $p > 3$. We use the identity

$$a_1|k-k'|^2 + a_2|k'|^2 = \frac{a_1 a_2}{a_1+a_2}|k|^2 + (a_1+a_2)\left|k' - \frac{a_1}{a_1+a_2}k\right|^2, \tag{15}$$

valid for arbitrary k, k'.

It follows easily from (14) that C_p grow no faster than exponentially (see Section 3), $C_p \leqslant b_1 b_2^p$ for some constants $b_1, b_2 < \infty$ depending on α.

Corollary. If $At^{\epsilon/2} < b_2^{-1}$ then the series (7) converges for every $k \in R^3 \setminus 0$.

It is interesting to remark that all but one of the terms of the series (7) have finite energy and enstrophy.

Other expansions for the NSS which are formal can be found in the monographs.[4,5] General approach to the existence problem for the NSS is discussed in ref. 3.

2. PROOF OF THE MAIN THEOREM

The proof goes by induction. First we derive the needed inequalities for $g_1(\tilde{k}, s)$. Using (11), (15) and the inequality $|c(k,0)| \leqslant 1$ we can write

$$|g_1(\tilde{k}, s)| \leqslant |\tilde{k}| \int_{R^3} \frac{e^{-|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha \cdot |\tilde{k}'|^\alpha} = |\tilde{k}|\, e^{-\frac{1}{2}|\tilde{k}|^2} \int_{R^3} \frac{e^{-2|\tilde{k}'-\frac{1}{2}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha \cdot |\tilde{k}'|^\alpha}. \tag{16}$$

For the last integral we have simple estimates:

(1) if $|\tilde{k}| \leqslant 1$ then

$$\int_{R^3} \frac{e^{-2|\tilde{k}' - \frac{1}{2}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k} - \tilde{k}'|^\alpha \cdot |\tilde{k}'|^\alpha} \leqslant \frac{B_1}{|\tilde{k}|^{2\alpha - 3}}$$

and therefore

$$|g_1(\tilde{k}, s)| \leqslant \frac{B_1}{|\tilde{k}|^{2\epsilon}} e^{-\frac{1}{2}|\tilde{k}|^2}; \tag{17}$$

(2) if $|\tilde{k}| \geqslant 1$ then

$$\int_{R^3} \frac{e^{-2|\tilde{k}' - \frac{1}{2}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k} - \tilde{k}'|^\alpha \cdot |\tilde{k}'|^\alpha} \leqslant \frac{B_2}{|\tilde{k}|^{2\alpha}}$$

and

$$|g_1(\tilde{k}, s)| \leqslant \frac{B_2}{|\tilde{k}|^{2\alpha - 1}} e^{-\frac{1}{2}|\tilde{k}|^2}. \tag{18}$$

Thus

$$|g_1(\tilde{k}, s)| \leqslant B_3 e^{-\frac{1}{2}|\tilde{k}|^2} f_1(|\tilde{k}|), \tag{19}$$

where

$$f_1(x) = \begin{cases} x^{-2\epsilon} & \text{if } 0 < x < 1, \\ x^{-(2\alpha - 1)} & \text{if } x \geqslant 1. \end{cases}$$

In the same manner we estimate $|g_2(k, s)|$. Using (12) and (15) we write

$$\begin{aligned} &|g_2(\tilde{k}, s)| \\ &\leqslant B_3|\tilde{k}| \left[\int_0^1 \tilde{s}_1^\epsilon \, d\tilde{s}_1 \cdot \int_{R^3} \frac{f_1(|\tilde{k} - \tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{1}{2}|\tilde{k} - \tilde{k}'|^2 \tilde{s}_1 - (1 - \tilde{s}_1)|\tilde{k} - \tilde{k}'|^2 - |\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha} \right. \\ &\qquad \left. + \int_0^1 \tilde{s}_2^\epsilon \, d\tilde{s}_2 \int_{R^3} \frac{f_1(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-|\tilde{k} - \tilde{k}'|^2 - (1 - \tilde{s}_2)|\tilde{k}'|^2 - \frac{1}{2}\tilde{s}_2 \cdot |\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k} - \tilde{k}'|^\alpha} \right] \end{aligned}$$

$$\leqslant B_3|\tilde{k}|\, e^{-\frac{|\tilde{k}|^2}{3}} \left[\int_0^1 \tilde{s}_1^{\epsilon}\, d\tilde{s}_1 \int_{R^3} \frac{f_1(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{3}{2}|\tilde{k}'-\frac{1}{3}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}} + \int_0^1 \tilde{s}_2^{\epsilon}\, d\tilde{s}_2 \int_{R^3} \frac{f_1(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-\frac{3}{2}|\tilde{k}'-\frac{1}{3}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^{\alpha}}\right].$$

We shall estimate only the first integral in the square brackets, the second one can be estimated in the same way. Again, consider two cases.

(1) $|\tilde{k}|\leqslant 1$. It is easy to see that

$$\int_0^1 \tilde{s}_1^{\epsilon}\, d\tilde{s}_1 \int_{R^3} \frac{f_1(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{3}{2}|\tilde{k}'-\frac{2}{3}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}} \leqslant B_4$$

if ϵ is sufficiently small.

(2) $|\tilde{k}|\geqslant 1$. In this case we shall change the order of integration and consider

$$\int_{R^3} \frac{e^{-\frac{3}{2}|\tilde{k}'-\frac{1}{3}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}} \left[\frac{1}{|\tilde{k}-\tilde{k}'|^{2\epsilon}} \int_0^{\min\left(1,\frac{1}{|\tilde{k}-\tilde{k}'|^2}\right)} d\tilde{s}_1 + \frac{1}{|\tilde{k}-\tilde{k}'|^{3+2\epsilon}} \int_{\min\left(1,\frac{1}{|\tilde{k}-\tilde{k}'|^2}\right)}^1 \frac{1}{\tilde{s}_1^{3+\epsilon}}\, d\tilde{s}_1\right]$$

$$= \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{e^{-\frac{3}{2}|\tilde{k}'-\frac{1}{3}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}\cdot|\tilde{k}-\tilde{k}'|^{2+2\epsilon}} + \int_{|\tilde{k}-\tilde{k}'|\leqslant 1} \frac{e^{-\frac{3}{2}|\tilde{k}'-\frac{1}{3}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}\,|\tilde{k}-\tilde{k}'|^{2\epsilon}}$$

$$\leqslant \frac{B_5}{|\tilde{k}|^{4+3\epsilon}} + B_6\, e^{-\frac{1}{6}|\tilde{k}|^2} \leqslant \frac{B_7}{|\tilde{k}|^{4+3\epsilon}}$$

and we write

$$|g_2(\tilde{k},s)| \leqslant B_9\, f_2(|\tilde{k}|)\, e^{-\frac{|\tilde{k}|^2}{3}}, \qquad (20)$$

where

$$f_2(x) = \begin{cases} x & \text{if } 0\leqslant x\leqslant 1,\\ \frac{1}{x^{3+3\epsilon}} & \text{if } x\geqslant 1.\end{cases}$$

Consider now g_3. The estimates which we shall derive here will be enough general so that later using the induction we shall be able to analyze g_p, $p > 3$.

It follows from (13) that

$$g_3(\tilde{k}, s) = i\Bigg[\int_0^1 \tilde{s}_1^{\frac{3\epsilon}{2}}\, d\tilde{s}_1$$

$$\int_{R^3} \frac{\langle \tilde{k}, g_2((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\tilde{s}_1)\rangle \cdot P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}}, 0\right) e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2 - |\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha}$$

$$+ \int_0^1 \tilde{s}_2^{\frac{3\epsilon}{2}}\, d\tilde{s}_2 \int_{R^3} \frac{\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}}, 0\right)\rangle P_{\tilde{k}} g_2\left(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\right) \cdot e^{-|\tilde{k}-\tilde{k}'|^2 - (1-\tilde{s}_2)|\tilde{k}'|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}$$

$$+ \int_0^1 \tilde{s}_1^\epsilon\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^\epsilon\, d\tilde{s}_2 \int_{R^3} \langle \tilde{k}, g_1((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\cdot\tilde{s}_1)\rangle P_{\tilde{k}} g_1(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2)$$

$$\cdot\, e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2 - (1-\tilde{s}_2)|\tilde{k}'|^2}\, d\tilde{k}'\Bigg]$$

and

$$|g_3(\tilde{k}, s)|$$

$$\leqslant |\tilde{k}| \Bigg[B_9 \Bigg(\int_0^1 \tilde{s}_1^{\frac{3\epsilon}{2}}\, d\tilde{s}_1 \int_{R^3} \frac{f_2(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1}) \cdot e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2 - \frac{\tilde{s}_1}{3}|\tilde{k}-\tilde{k}'|^2 - |k'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha}$$

$$+ \int_0^1 \tilde{s}_2^{\frac{3\epsilon}{2}}\, d\tilde{s}_2 \int_{R^3} \frac{f_2(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-|\tilde{k}-\tilde{k}'|^2 - (1-\tilde{s}_2)|\tilde{k}'|^2 - \tilde{s}_2 \frac{|\tilde{k}'|^2}{3}}\, dk'}{|\tilde{k}-\tilde{k}'|^\alpha} \Bigg)$$

$$+ B_3 \int_0^1 \tilde{s}_1^\epsilon\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^\epsilon\, d\tilde{s}_2 \int_{R^3} f_1(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1}) \cdot f_1(|\tilde{k}'|\sqrt{\tilde{s}_2})$$

$$\cdot\, e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2 - \frac{\tilde{s}_1}{2}|\tilde{k}-\tilde{k}'|^2 - (1-\tilde{s}_2)|\tilde{k}'|^2 - \frac{\tilde{s}_2}{2}\cdot|\tilde{k}'|^2} dk' \Bigg]$$

$$\leqslant B_{10} \cdot |\tilde{k}| \cdot e^{-\frac{|\tilde{k}|^2}{4}} \cdot \Bigg[\int_0^1 \tilde{s}_1^{\frac{3\epsilon}{2}}\, d\tilde{s}_1 \int_{R^3} \frac{f_2(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{4}{3}|\tilde{k}'-\frac{1}{4}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha}$$

$$+ \int_0^1 \tilde{s}_2^{\frac{3\epsilon}{2}}\, d\tilde{s}_2 \int_{R^3} \frac{f_2(|\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{4}{3}|\tilde{k}'-\frac{3}{4}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}$$

$$+ \int_0^1 \tilde{s}_1^\epsilon\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^\epsilon\, d\tilde{s}_2 \int_{R^3} f_1(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, f_1(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\, d\tilde{k}' \Bigg].$$

Again we consider two cases.
Case 1. $|\tilde{k}| \leqslant 1$. This is a simple case.
We have to estimate

$$\int_0^1 \tilde{s}_1^{\frac{3\epsilon}{2}} \, d\tilde{s}_1 \int_{R^3} \frac{f_2(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{4}{3}|\tilde{k}'-\frac{3}{4}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^\alpha} + \int_0^1 \tilde{s}_2^{\frac{3\epsilon}{2}} \, d\tilde{s}_2 \int_{R^3} \frac{f_2(|\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{4}{3}|\tilde{k}'-\frac{1}{4}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha} + \int_0^1 \tilde{s}_1^{\epsilon}\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^{\epsilon}\, d\tilde{s}_2 \int_{R^3} f_1(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, f_1(|\tilde{k}'|\sqrt{\tilde{s}_2}) \cdot e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\, d\tilde{k}'.$$

It is easy to see that the last expression is less than some constant depending on ϵ, i.e., on α which we shall denote by B_{11}. This estimate is enough for our purposes.

Case 2. $|\tilde{k}| \geqslant 1$. First we estimate

$$I_1 = \int_0^1 \tilde{s}_1^{\frac{3\epsilon}{2}} \, d\tilde{s}_1 \int_{R^3} \frac{f_2(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{4}{3}|\tilde{k}'-\frac{1}{4}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^\alpha}.$$

We have

$$I_1 = \int_{|\tilde{k}-\tilde{k}'|\leqslant 1} \frac{e^{-\frac{4}{3}|\tilde{k}'-\frac{1}{4}\tilde{k}|^2}\, d\tilde{k}' |\tilde{k}-\tilde{k}'|}{|\tilde{k}'|^\alpha} \int_0^1 \tilde{s}_1^{\frac{3\epsilon}{2}+\frac{1}{2}}\, d\tilde{s}_1 + \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{e^{-\frac{4}{3}|\tilde{k}'-\frac{1}{4}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^\alpha} \left[|\tilde{k}-\tilde{k}'| \int_0^{\frac{1}{|\tilde{k}-\tilde{k}'|^2}} \tilde{s}_1^{\frac{3\epsilon}{2}+\frac{1}{2}}\, d\tilde{s}_1 + \frac{1}{|\tilde{k}-\tilde{k}'|^{3+3\epsilon}} \int_{\frac{1}{|\tilde{k}-\tilde{k}'|^2}}^1 \tilde{s}_1^{-\frac{3}{2}}\, d\tilde{s}_1 \right] \leqslant \frac{B_{11}}{|\tilde{k}|^\alpha} + B_{12} \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{e^{-\frac{4}{3}|\tilde{k}'-\frac{1}{4}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^\alpha \cdot |\tilde{k}-\tilde{k}'|^{2+3\epsilon}} \leqslant \frac{B_{13}}{|\tilde{k}|^\alpha}.$$

The integral

$$I_2 = \int_0^1 \tilde{s}_2^{\frac{3\epsilon}{2}} \, d\tilde{s}_2 \int_{R^3} \frac{f_2(|\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{4}{3}|\tilde{k}'-\frac{3}{4}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}$$

is estimated in the same way.

It remains to estimate

$$I_3 = \int_0^1 \tilde{s}_1^{\epsilon}\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^{\epsilon}\, d\tilde{s}_2 \int_{R^3} f_1(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1}) \cdot f_1(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\, d\tilde{k}' .$$

Consider four cases.

(a_1) $|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1} \leqslant 1, |\tilde{k}'|\sqrt{\tilde{s}_2} \leqslant 1$ or

$$0 \leqslant \tilde{s}_1 \leqslant \min\left(1, \frac{1}{|\tilde{k}-\tilde{k}'|^2}\right), \qquad 0 \leqslant \tilde{s}_2 \leqslant \min\left(1, \frac{1}{|\tilde{k}'|^2}\right).$$

Using (19) we can write

$$\int_{R^3} |\tilde{k}-\tilde{k}'|^{-2\epsilon} \cdot |\tilde{k}'|^{-2\epsilon}\, e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\, d\tilde{k}' \int_0^{\min\left(1, \frac{1}{|\tilde{k}-\tilde{k}'|^2}\right)} d\tilde{s}_1 \,.$$

$$\int_0^{\min\left(1, \frac{1}{|\tilde{k}-\tilde{k}'|^2}\right)} d\tilde{s}_2 \leqslant \int_{R^3} |\tilde{k}-\tilde{k}'|^{-2-2\epsilon} \cdot |\tilde{k}'|^{-2-2\epsilon}\, e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\, d\tilde{k}' \leqslant \frac{B_{14}}{|\tilde{k}|^{4+4\epsilon}} .$$

(a_2) $|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1} \geqslant 1,\ |\tilde{k}'|\sqrt{\tilde{s}_2} \leqslant 1$ or

$$\min\left(1, \frac{1}{|\tilde{k}-\tilde{k}'|^2}\right) \leqslant \tilde{s}_1 \leqslant 1, \quad 0 \leqslant \tilde{s}_2 \leqslant \frac{1}{|\tilde{k}'|^2}.$$

In this case, only the integration over the domain $|\tilde{k}'-\tilde{k}| \geqslant 1$ has meaning. We can write using (19)

$$\int_{|\tilde{k}'-\tilde{k}|\geqslant 1} e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\, d\tilde{k} \cdot \frac{1}{|\tilde{k}-\tilde{k}'|^{3+2\epsilon}\,|\tilde{k}'|^{2\epsilon}} \cdot \int_{\frac{1}{|\tilde{k}-\tilde{k}'|^2}}^1 \frac{d\tilde{s}_1}{\tilde{s}_1^{3/2}} \cdot \int_0^{\frac{1}{|\tilde{k}'|^2}} d\tilde{s}_2$$

$$\leqslant B_{15} \cdot \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{|\tilde{k}'| \cdot |\tilde{k}-\tilde{k}'|\, e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^{3+2\epsilon} \cdot |\tilde{k}'|^{2+2\epsilon}} \leqslant \frac{B_{16}}{|\tilde{k}|^{4+4\epsilon}} .$$

(a_3) $|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1} \leqslant 1,\ |\tilde{k}'|\sqrt{\tilde{s}_2} \geqslant 1$. The estimates are the same as in (a_2).

(a$_4$) $|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1}\geqslant 1,\ |\tilde{k}'|\sqrt{\tilde{s}_2}>1$ or

$$\min\left(\frac{1}{|\tilde{k}-\tilde{k}'|^2},1\right)\leqslant \tilde{s}_1\leqslant 1,\quad \min\left(\frac{1}{|\tilde{k}'|^2},1\right)\leqslant \tilde{s}_2\leqslant 1\,.$$

The integration is meaningful if $|\tilde{k}-\tilde{k}'|\geqslant 1, |\tilde{k}'|\geqslant 1$. We can write using (13)

$$\int_{\substack{|\tilde{k}-\tilde{k}'|\geqslant|\\ |\tilde{k}|\geqslant 1}} \frac{e^{-|\tilde{k}'-\frac{1}{2}\tilde{k}|^2}\,d\tilde{k}'}{|\tilde{k}'-\tilde{k}|^{3+2\epsilon}\cdot|\tilde{k}'|^{3+2\epsilon}}\int_{\frac{1}{|\tilde{k}-\tilde{k}'|^2}}^{1}\tilde{s}_1^{-\frac{3}{2}}\,d\tilde{s}_1\int_{\frac{1}{|\tilde{k}'|^2}}^{1}\tilde{s}_2^{-\frac{3}{2}}\,d\tilde{s}_2\leqslant\frac{B'_{17}}{|\tilde{k}|^{4+4\epsilon}}\,.$$

Collecting all the estimates we can write

$$|g_3(\tilde{k},s)|\leqslant B_{18}\cdot f_3(|\tilde{k}|)\,e^{-\frac{|\tilde{k}|^2}{4}},\tag{21}$$

where

$$f_3(x)=\begin{cases} x & 0\leqslant x\leqslant 1,\\ \frac{1}{x^{1+\epsilon}} & x\geqslant 1\,. \end{cases}$$

Now we consider general $p>3$ (see (13)) and use the induction. Our inductive assumption says: for $2<q<p$

$$|g_q(\tilde{k},s)|\leqslant C_q\,f(|\tilde{k}|)\,e^{-\frac{|\tilde{k}|^2}{p+1}},$$

where C_q are some constants,

$$f(x)=f_q(x)=\begin{cases} x & 0\leqslant x\leqslant 1,\\ x^{-1} & x\geqslant 1. \end{cases}$$

For $q=1,2,3$ the inductive assumptions were checked. We have (see (13))

$$\begin{aligned}
&|g_p(\tilde{k},s)| \\
&\leqslant |\tilde{k}|\Bigg[\int_0^1 \tilde{s}_1^{\frac{p\epsilon}{2}}\,d\tilde{s}_1\cdot\int_{R^3}\frac{|g_{p-1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1},ss_1)|\,e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2}\,d\tilde{k}'}{|\tilde{k}'|^\alpha} \\
&+\int_0^1 \tilde{s}_2^{\frac{p\epsilon}{2}}\,d\tilde{s}_2\int_{R^3}\frac{|g_{p-1}(\tilde{k}'\sqrt{\tilde{s}_2},s\tilde{s}_2)|e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}\,d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha} \\
&+\sum_{\substack{p_1,p_2\geqslant 1\\ p_1+p_2=p-1}}\int_0^1 \tilde{s}_1^{\frac{(p_1+1)\epsilon}{2}}\,d\tilde{s}_1\int_0^1 \tilde{s}_2^{\frac{(p_2+1)}{2}\epsilon}\,d\tilde{s}_2 \\
&\cdot\int_{R^3}|g_{p_1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1},\ s\tilde{s}_1)|\,|g_{p_2}(\tilde{k}'\sqrt{\tilde{s}_2},\ s\tilde{s}_2)|\,e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}\,d\tilde{k}'\Bigg].
\end{aligned}$$

Using the inductive assumption and (15) we write

$$\begin{aligned}
&|g_p(\tilde{k},s)| \\
&\leqslant |\tilde{k}|\Bigg[C_{p-1}\int_0^1 \tilde{s}_1^{\frac{p\epsilon}{2}}\,d\tilde{s}_1\int_{R^3}\frac{e^{-\frac{\tilde{s}_1|\tilde{k}-\tilde{k}'|^2}{p}-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2}f_{p-1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\,d\tilde{k}'}{|\tilde{k}'|^\alpha} \\
&+C_{p-1}\int_0^1 \tilde{s}_2^{\frac{p\epsilon}{2}}\,d\tilde{s}_2\int_{R^3}\frac{e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_1)|\tilde{k}'|^2-\frac{\tilde{s}_2}{p}|\tilde{k}'|^2}f_{p-1}(|\tilde{k}'|\sqrt{\tilde{s}_1})\,d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha} \\
&+\sum_{\substack{p_1,p_2\geqslant 1\\ p_1+p_2=p-1}}C_{p_1}C_{p_2}\int_0^1 \tilde{s}_1^{\frac{(p_1+1)}{2}\epsilon}\,d\tilde{s}_1\int_0^1 \tilde{s}_2^{\frac{(p_2+1)}{2}\epsilon}\int_{R^3}d\tilde{k}'\,f_{p_1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})f_{p_2}(|\tilde{k}'|\sqrt{\tilde{s}_2}) \\
&\cdot\ e^{-\frac{\tilde{s}_1|\tilde{k}-\tilde{k}'|^2}{p_1+1}-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-\frac{\tilde{s}_2|\tilde{k}'|^2}{p_2+1}-(1-\tilde{s}_2)|\tilde{k}'|^2}\Bigg] \\
&\leqslant |\tilde{k}|\Bigg[C_{p-1}\int_0^1 \tilde{s}_1^{\frac{p\epsilon}{2}}\,d\tilde{s}_1\int_{R^3}\frac{f_{p-1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\cdot e^{-\frac{|\tilde{k}-\tilde{k}'|^2}{p}-|\tilde{k}'|^2}\,d\tilde{k}'}{|\tilde{k}'|^\alpha} \\
&+C_{p-1}\int_0^1 \tilde{s}_2^{\frac{p\epsilon}{2}}\,d\tilde{s}_2\int_{R^3}\frac{e^{-|\tilde{k}-\tilde{k}'|^2-\frac{|\tilde{k}'|^2}{p}}f_{p-1}(|\tilde{k}'|\sqrt{\tilde{s}_1})d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha} \\
&+\sum_{\substack{p_1\geqslant 1,p_2\geqslant 1\\ p_1+p_2=p-1}}C_{p_1}\cdot C_{p_2}\int_0^1 \tilde{s}_1^{\frac{(p_1+1)}{2}\epsilon}\,d\tilde{s}_1\int_0^1 \tilde{s}_2^{\frac{(p_2+1)}{2}\epsilon}\,d\tilde{s}_2 \\
&\cdot\int_{R^3}f_{p_1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\,f_{p_2}(|\tilde{k}'|\sqrt{\tilde{s}_2})\,e^{-\frac{|\tilde{k}-\tilde{k}'|^2}{p_1+1}-\frac{|\tilde{k}'|^2}{p_2+1}}\,d\tilde{k}'\Bigg].
\end{aligned}$$

The estimates below are done for $p_1, p_2 > 1$. The case $p_1 = 1$ or $p_2 = 1$ requires trivial changes. We continue

$$
\begin{aligned}
&|g_p(\tilde{k}, s)| \\
&\leqslant |\tilde{k}|\, e^{-\frac{|\tilde{k}|^2}{p+1}} \Bigg[C_{p-1} \int_0^1 \tilde{s}_1^{\frac{p\epsilon}{2}}\, d\tilde{s}_1 \int_{R^3} \frac{f_{p-1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha} \\
&+ C_{p-1} \int_0^1 \tilde{s}_2^{\frac{p\epsilon}{2}}\, d\tilde{s}_2 \int_{R^3} \frac{f_{p-1}(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-\frac{p+1}{p}|\tilde{k}'-\frac{p}{p+1}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha} \\
&+ \sum_{\substack{p_1 \geqslant 1, p_2 \geqslant 1 \\ p_1+p_2=p-1}} C_{p_1} \cdot C_{p_2} \int_0^1 \tilde{s}_1^{\frac{(p_1+1)}{2}\epsilon}\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^{\frac{(p_2+1)}{2}\epsilon}\, d\tilde{s}_2 \\
&\cdot \int_{R^3} f_{p_1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, f_{p_2}(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-\frac{p+1}{(p_1+1)(p_2+1)}|\tilde{k}'-\frac{p_2+1}{p+1}\tilde{k}|^2}\, d\tilde{k}' \Bigg]. \qquad (22)
\end{aligned}
$$

As before, consider two cases.

Case 1. $|\tilde{k}| \leqslant 1$. The function f is bounded from above by 1. Therefore the first two integrals in the square brackets in (22) are bounded from above by some constant B_{19}. Concerning the last sum it is not more than $B_{20}\left(\frac{(p_1+1)(p_2+1)}{p+1}\right)^{1/2}$.

Finally we can write

$$
|g_p(\tilde{k}, s)| \leqslant |\tilde{k}|\, e^{-\frac{|k|^2}{p+1}} \cdot B_{21} \cdot \sum_{\substack{p_1, p_2 \geqslant 0 \\ p_1+p_2=p-1}} C_{p_1} \cdot C_{p_2} \left(\frac{(p_1+1)(p_2+1)}{p+1}\right)^{1/2}. \qquad (23)
$$

This is the inequality which we need.

Case 2. $|\tilde{k}| \geqslant 1$. First we estimate the integral

$$
I_1 = \int_0^1 \tilde{s}_1^{\frac{p\epsilon}{2}}\, d\tilde{s}_1 \int_{R^3} \frac{f_{p-1}(|\tilde{k}-\tilde{k}'|\sqrt{\tilde{s}_1})\, e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha}
$$

Using the form of f_{p-1} and changing the order of integration we can write

$$
\begin{aligned}
I_1 = & \int_{|\tilde{k}-\tilde{k}'|\leqslant 1} \frac{|\tilde{k}-\tilde{k}'|\, e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}} \int_0^1 \tilde{s}_1^{\frac{1}{2}+\frac{p\epsilon}{2}}\, ds_1 \\
& + \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{|\tilde{k}-\tilde{k}'|\, e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}} \int_0^{\frac{1}{|\tilde{k}-\tilde{k}'|^2}} \tilde{s}_1^{\frac{1}{2}+\frac{p\epsilon}{2}}\, d\tilde{s}_1 \\
& + \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2}}{|\tilde{k}'|^{\alpha}\cdot|\tilde{k}-\tilde{k}'|} \int_{\frac{1}{|\tilde{k}-\tilde{k}'|^2}}^1 \tilde{s}_1^{-\frac{1}{2}+\frac{p\epsilon}{2}}\, d\tilde{s}_1 \\
\leqslant & \frac{2}{3+p\epsilon} \int_{|\tilde{k}-\tilde{k}'|\leqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2}|\tilde{k}-\tilde{k}'|\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}} \\
& + \frac{2}{3+p\epsilon} \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^{2+p\epsilon}\cdot|\tilde{k}'|^{\alpha}} \\
& + \frac{2}{1+p\epsilon} \int_{|\tilde{k}-\tilde{k}'|\geqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}'-\frac{1}{p+1}\tilde{k}|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}\cdot|\tilde{k}-\tilde{k}'|} = I_1^{(1)} + I_1^{(2)} + I_1^{(3)}.
\end{aligned}
$$

The first integral satisfies the obvious estimate:

$$I_1^{(1)} \leqslant B_{22}\, e^{-\frac{|\tilde{k}|^2}{2}}.$$

In the integral $I_1^{(2)}$ make the shift $\tilde{k}' = \frac{\tilde{k}}{p+1} + \tilde{k}''$. Then

$$I_1^{(2)} = \frac{2}{3+p\epsilon} \int_{|\tilde{k}\frac{p}{p+1}-\tilde{k}''|\geqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}''|^2}\, d\tilde{k}''}{|\tilde{k}\frac{p}{p+1} - \tilde{k}''|^{2+p\epsilon}\cdot|\frac{\tilde{k}}{p+1} + \tilde{k}''|^{\alpha}}.$$

Now it is clear that for $I_1^{(2)}$ we can write

$$I_1^{(2)} \leqslant \frac{B_{23}}{|\tilde{k}|^{2+p\epsilon}}.$$

The estimation of the third integral is the most difficult. Again we write $\tilde{k}'' = \tilde{k}' - \frac{1}{p+1}\tilde{k}$.

Then

$$I_1^{(3)} = \frac{2}{1+p\epsilon} \int_{|\frac{p}{p+1}\tilde{k}-\tilde{k}''|\geqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}''|^2} d\tilde{k}''}{|\frac{\tilde{k}}{p+1}+\tilde{k}''|^\alpha |\frac{p}{p+1}\tilde{k}-\tilde{k}''|}.$$

If p is so large that $\frac{|\tilde{k}|}{p+1} \leqslant 1$ then

$$\int_{|\frac{p}{p+1}\tilde{k}-\tilde{k}''|\geqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}''|^2} d\tilde{k}''}{|\frac{\tilde{k}}{p+1}+\tilde{k}''|^\alpha |\frac{p}{p+1}\tilde{k}-\tilde{k}''|} \leqslant \frac{B_{24}}{|\tilde{k}|}$$

and

$$I_1^{(3)} \leqslant \frac{2B_{24}}{(1+p\epsilon)|\tilde{k}|} \leqslant \frac{2B_{24}}{\epsilon} \cdot \frac{1}{p} \cdot \frac{1}{|\tilde{k}|} \leqslant \frac{B_{25}}{|\tilde{k}|^2}.$$

If $\frac{|\tilde{k}|}{p+1} \geqslant 1$ then we take into account the factor $\frac{1}{|\frac{\tilde{k}}{p+1}+\tilde{k}''|^\alpha}$ and write

$$\int_{|\frac{p}{p+1}\tilde{k}-\tilde{k}''|\geqslant 1} \frac{e^{-\frac{p+1}{p}|\tilde{k}''|^2} d\tilde{k}''}{|\frac{\tilde{k}}{p+1}+\tilde{k}''|^\alpha \cdot |\frac{p}{p+1}\tilde{k}-\tilde{k}''|} \leqslant \frac{B_{26}\cdot p^\alpha}{|\tilde{k}|^\alpha \cdot |\tilde{k}|}.$$

Thus

$$I_1^{(3)} \leqslant \frac{B_{27}\cdot p^{\alpha-1}}{|\tilde{k}|^\alpha \cdot |\tilde{k}|} \leqslant \frac{B_{28}\cdot |\tilde{k}|^{\alpha-1}}{|\tilde{k}|^\alpha \cdot |\tilde{k}|} = \frac{B_{28}}{|\tilde{k}|^2}.$$

Finally we can write

$$I_1 \leqslant \frac{B_{29}}{|\tilde{k}|^2}. \tag{24}$$

This is the estimate which we need.

Consider the integral

$$I_2 = \int_0^1 \tilde{s}_2^{\frac{p\epsilon}{2}} d\tilde{s}_2 \int_{R^3} \frac{f_{p-1}(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-\frac{p+1}{p}|\tilde{k}'-\frac{p}{p+1}\tilde{k}|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}.$$

It is reduced to I_1 with the help of the change of variables $\tilde{k}-\tilde{k}'=\tilde{k}''$. Let

$$I_{p_1,p_2} = \int_0^1 \tilde{s}_1^{\frac{p_1+1}{2}} d\tilde{s}_1 \int_0^1 \tilde{s}_2^{\frac{p_2+1}{2}} \int_{R^3} f_{p_1}$$
$$\cdot (|\tilde{k} - \tilde{k}'|\sqrt{\tilde{s}_1})\, f_{p_2}(|\tilde{k}'|\sqrt{\tilde{s}_2})\, e^{-\frac{(p+1)|\tilde{k}' - \frac{p_2+1}{p+1}\tilde{k}|^2}{(p_1+1)(p_2+1)}} d\tilde{k}' .$$

For $p_1, p_2 > 1$ we can write $I_{p_1,p_2} = \sum_{j=1}^4 I^{(j)}_{p_1,p_2}$ where

$$I^{(1)}_{p_1,p_2} = \int_{|\tilde{k}-\tilde{k}'| \leqslant 1, |\tilde{k}'| \leqslant 1} e^{-\frac{(p+1)}{(p_1+1)(p_2+1)}|\tilde{k}' - \frac{p_2+1}{p+1}\tilde{k}|^2} \cdot |\tilde{k} - \tilde{k}'| \cdot |\tilde{k}'| d\tilde{k}'$$
$$\cdot \int_0^1 \tilde{s}_1^{\frac{1}{2}+\frac{p_1+1}{2}\epsilon} d\tilde{s}_1 \int_0^1 \tilde{s}_2^{\frac{1}{2}+\frac{p_1+1}{2}\epsilon} d\tilde{s}_2 .$$

This integral is different from zero only if $|\tilde{k}| \leqslant 2$ and in this case the estimates are the same as in the Case 1. In the estimates below we assume that $|\tilde{k}| \geqslant 4$. The case $|\tilde{k}| \leqslant 4$ can be treated in the same way as $|\tilde{k}| \leqslant 1$. The cases $p_1 = 1$ or $p_2 = 1$ need trivial changes.

$$I^{(2)}_{p_1,p_2} = \int_{|\tilde{k}-\tilde{k}'| \geqslant 1, |\tilde{k}'| \leqslant 1} d\tilde{k}' e^{-\frac{(p+1)}{(p_1+1)(p_2+1)}|\tilde{k}' - \frac{p_2+1}{p+1}\tilde{k}|^2} \left[\frac{|\tilde{k}'|}{|\tilde{k} - \tilde{k}'|} \int_{\frac{1}{|\tilde{k}-\tilde{k}'|^2}}^1 \tilde{s}_1^{-\frac{1}{2}+\frac{p_1+1}{2}\epsilon} d\tilde{s}_1 \right.$$
$$\left. \cdot \int_0^1 \tilde{s}_2^{\frac{1}{2}+\frac{p_2+1}{2}\epsilon} d\tilde{s}_2 + |\tilde{k} - \tilde{k}'| \cdot |\tilde{k}'| \int_0^{\frac{1}{|\tilde{k}-\tilde{k}'|^2}} \tilde{s}_1^{\frac{1}{2}+\frac{p_1+1}{2}\epsilon} d\tilde{s}_1 \int_0^1 \tilde{s}_2^{\frac{1}{2}+\frac{p_2+1}{2}\epsilon} d\tilde{s}_2 \right]$$
$$\leqslant B_{30} \int_{|\tilde{k}-\tilde{k}'| \geqslant 1, |\tilde{k}'| \leqslant 1} e^{-\frac{(p+1)}{(p_1+1)(p_2+1)} \cdot |\tilde{k}' - \frac{p_2+1}{p+1}\tilde{k}|^2} d\tilde{k}'$$
$$\cdot \left[\frac{1}{|\tilde{k}| \cdot (1 + (p_1+1)\epsilon)(3 + (p_2+1)\epsilon)} \right.$$
$$\left. + \frac{1}{|\tilde{k} - \tilde{k}'|^{2+(p_1+1)\epsilon}(3 + (p_1+1)\epsilon)(\frac{3}{2} + (p_2+1)\epsilon)} \right].$$

Assume that $p_2 \leqslant \frac{1}{2}(p-1)$, i.e. $p_2 \leqslant p_1$. If $\frac{p_2+1}{p+1} \cdot |\tilde{k}| \leqslant 2$ then

$$I^{(2)}_{p_1,p_2} \leqslant \frac{B_{31}}{|\tilde{k}| \cdot (p_1+1)(p_2+1)} \leqslant \frac{B_{32}}{|\tilde{k}| \cdot (p+1)(p_2+1)} \leqslant \frac{2 \cdot B_{32}}{|\tilde{k}|^2 (p_2+1)^2} \leqslant \frac{B_{33}}{|\tilde{k}|^2} .$$

If $\frac{p_2+1}{p+1}|\tilde{k}| \geqslant 2$ then $e^{-\frac{(p+1)}{(p_1+1)(p_2+1)}|\tilde{k}'-\frac{p_2+1}{p+1}\tilde{k}|^2} \leqslant e^{-\frac{(p_2+1)|\tilde{k}|^2}{4(p_1+1)(p+1)}}$. Denote $z_1 = \frac{(p_2+1)|\tilde{k}|}{p+1}$.
Then

$$
\begin{aligned}
I^{(2)}_{p_1,p_2} &\leqslant B_{34}\, e^{-\frac{1}{8}z_1^2} \cdot \left[\frac{1}{|\tilde{k}|} \cdot \frac{z_1}{|\tilde{k}|(p_2+1)^{3/2}} + \frac{B_{35}}{|\tilde{k}|^{2+(p_1+1)\epsilon}}\right] \\
&\leqslant B_{35}\, \frac{e^{-\frac{1}{8}z_1^2}\cdot z_1}{|\tilde{k}|^2} \leqslant \frac{B_{36}}{|\tilde{k}|^2}.
\end{aligned}
$$

If $p_2 \geqslant \frac{1}{2}(p-1)$ then

$$
\begin{aligned}
&I^{(2)}_{p_1,p_2} \\
&\leqslant B_{37} e^{-\frac{(p_2+1)}{(p_1+1)(p+1)}|\tilde{k}|^2} \cdot \left[\frac{1}{|\tilde{k}|(p_1+1)(p_2+1)} + \frac{1}{|\tilde{k}|^{2+(p_1+1)\epsilon}(p_1+1)\cdot(p_2+1)}\right] \\
&\leqslant B_{37} e^{-\frac{|\tilde{k}|^2}{3(p_1+1)}} \cdot \frac{|\tilde{k}|}{\sqrt{p_1+1}}\left[\frac{1}{|\tilde{k}|^2\sqrt{p_1+1}\cdot p} + \frac{1}{|\tilde{k}|^{3+(p_1+1)\epsilon}\cdot\sqrt{p_1+1}\cdot(p_2+1)}\right] \\
&\leqslant \frac{B_{38}}{|\tilde{k}|^2}.
\end{aligned}
$$

It follows from all these estimates that $I^{(2)}_{p_1,p_2} \leqslant \frac{B_{39}}{|\tilde{k}|^2}$. This estimate is also valid if p_1 or p_2 equals 1.

This is the inequality which we need.

The integral

$$
\begin{aligned}
I^{(3)}_{p_1,p_2} = \int_{|\tilde{k}-\tilde{k}'|\leqslant 1, |\tilde{k}'|\geqslant 1} & e^{-\frac{(p+1)}{(p_1+1)(p_2+1)}|\tilde{k}'-\frac{p_2+1}{p+1}\tilde{k}|^2} \left[\frac{|\tilde{k}-\tilde{k}'|}{|\tilde{k}'|}\int_0^1 \tilde{s}_1^{\frac{1}{2}+\frac{p_1+1}{2}\epsilon}\, d\tilde{s}_1\right. \\
&\cdot \int_{\frac{1}{|\tilde{k}'|^2}}^1 \tilde{s}_2^{-\frac{1}{2}+\frac{p_2+1}{2}\epsilon}\, d\tilde{s}_2 + |\tilde{k}-\tilde{k}'|\,|\tilde{k}'| \cdot \int_0^1 \tilde{s}_1^{\frac{1}{2}+\frac{(p_1+1)}{2}\epsilon} d\tilde{s}_1 \\
&\left.\cdot \int_0^{\frac{1}{|\tilde{k}'|^2}} \tilde{s}_2^{\frac{1}{2}+\frac{p_2+1}{2}\epsilon}\, d\tilde{s}_2\right]
\end{aligned}
$$

can be estimated in the same way as $I^{(2)}_{p_1,p_2}$.

It remains to consider

$$
\begin{aligned}
I^{(4)}_{p_1,p_2} = & \int_{|\tilde{k}-\tilde{k}'|\geqslant 1,\,|\tilde{k}'|\geqslant 1} e^{-\frac{(p+1)}{(p_1+1)(p_2+1)}\,|\tilde{k}'-\frac{p_2+1}{p+1}\,\tilde{k}|^2}\,d\tilde{k}' \\
& \cdot\Bigg[|\tilde{k}-\tilde{k}'|\cdot|\tilde{k}'|\int_0^{\frac{1}{|\tilde{k}-\tilde{k}'|^2}} \tilde{s}_1^{\frac{1}{2}+\frac{p_1+1}{2}\epsilon}\,d\tilde{s}_1 \\
& \cdot\int_0^{\frac{1}{|\tilde{k}'|^2}} \tilde{s}_2^{\frac{1}{2}+\frac{p_2+1}{2}\epsilon}\,d\tilde{s}_2 + \frac{1}{|\tilde{k}-\tilde{k}'|}\cdot|\tilde{k}|\int_{\frac{1}{|\tilde{k}-\tilde{k}'|^2}}^1 \tilde{s}_1^{-\frac{1}{2}+\frac{p_1+1}{2}\epsilon}\,d\tilde{s}_1 \\
& \cdot\int_0^{\frac{1}{|\tilde{k}'|^2}} \tilde{s}_2^{\frac{1}{2}+\frac{p_2+1}{2}\epsilon}\,d\tilde{s}_2 + |\tilde{k}-\tilde{k}'|\cdot\frac{1}{|\tilde{k}'|}\int_0^{\frac{1}{|\tilde{k}-\tilde{k}'|^2}} \tilde{s}_1^{\frac{1}{2}+\frac{p_1+1}{2}\epsilon}\,d\tilde{s}_1 \\
& \cdot\int_{\frac{1}{|\tilde{k}'|^2}}^1 \tilde{s}_2^{-\frac{1}{2}+\frac{p_2+1}{2}\epsilon}\,d\tilde{s}_2 + \frac{1}{|\tilde{k}-\tilde{k}'|\cdot|\tilde{k}'|}\cdot\int_{\frac{1}{|\tilde{k}-\tilde{k}'|^2}}^1 \tilde{s}_1^{-\frac{1}{2}+\frac{p_1+1}{2}\epsilon}\,d\tilde{s}_1 \\
& \cdot\int_{\frac{1}{|\tilde{k}'|^2}}^1 \tilde{s}_2^{-\frac{1}{2}+\frac{p_2+1}{2}\epsilon}\,d\tilde{s}_2\Bigg] \leqslant \int_{|\tilde{k}-\tilde{k}'|\geqslant 1,|\tilde{k}|\geqslant 1} e^{-\frac{p+1}{(p_1+1)(p_2+1)}\cdot|\tilde{k}'-\frac{p_2+1}{p+1}\,\tilde{k}|^2}\,d\tilde{k}' \\
& \cdot\Bigg[\frac{1}{|\tilde{k}-\tilde{k}'|^{2+\frac{p_1+1}{2}\epsilon}\,|\tilde{k}'|^{2+\frac{p_2+1}{2}\epsilon}}\cdot\frac{2}{3+(p_1+1)\epsilon}\cdot\frac{2}{3+(p_2+1)\epsilon} \\
& \quad+\frac{1}{|\tilde{k}-\tilde{k}'|}\cdot\frac{1}{|\tilde{k}'|^{2+\frac{p_2+1}{2}\epsilon}}\cdot\frac{2}{1+(p_1+1)\epsilon}\cdot\frac{2}{3+(p_2+1)^{\epsilon}} \\
& \quad+\frac{1}{|\tilde{k}-\tilde{k}'|^{2+\frac{p_1+1}{2}\epsilon}}\cdot\frac{1}{|\tilde{k}'|}\cdot\frac{2}{3+(p_1+1)\epsilon}\cdot\frac{2}{1+(p_2+1)\epsilon} \\
& \quad+\frac{1}{|\tilde{k}-\tilde{k}'|\cdot|\tilde{k}'|}\cdot\frac{2}{1+(p_1+1)\epsilon}\cdot\frac{2}{1+(p_2+1)\epsilon}\Bigg] \\
& \leqslant \frac{B_{39}}{(p_1+1)(p_2+1)}\int_{|\tilde{k}-\tilde{k}'|\geqslant 1,\,|\tilde{k}'|\geqslant 1} \frac{e^{-\frac{(p+1)}{(p_1+1)(p_2+1)}\cdot|\tilde{k}'-\frac{p_2+1}{p+1}\,\tilde{k}|^2}\,d\tilde{k}'}{|\tilde{k}-\tilde{k}'|\cdot|\tilde{k}'|}.
\end{aligned}
\tag{25}
$$

Make the change of variables:

$$
\tilde{\tilde{k}}' = \sqrt{\frac{p+1}{(p_1+1)(p_2+1)}}\,\tilde{k}', \quad \tilde{\tilde{k}} = \sqrt{\frac{p+1}{(p_1+1)(p_2+1)}}\,\tilde{k}\,.
$$

Then the last expression takes the form

$$\frac{B_{39}}{\sqrt{(p+1)(p_1+1)(p_2+1)}}\int_{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\geqslant\gamma\,,|\tilde{\tilde{k}}'|\geqslant\gamma}\frac{e^{-|\tilde{\tilde{k}}'-\frac{p_2+1}{p+1}\tilde{\tilde{k}}|^2}\,d\tilde{\tilde{k}}'}{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\cdot|\tilde{\tilde{k}}'|},\tag{26}$$

where $\gamma=\sqrt{\frac{p+1}{(p_1+1)(p_2+1)}}$. Assume that $p_2\leqslant p_1$. Then $\frac{p_2+1}{p+1}\leqslant\frac{1}{2}$ and

$$\int_{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\geqslant\gamma\,,|\tilde{\tilde{k}}'|\geqslant\gamma}\frac{e^{-|\tilde{\tilde{k}}'-\frac{p_2+1}{p+1}\tilde{\tilde{k}}|^2}\,d\tilde{\tilde{k}}'}{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\cdot|\tilde{\tilde{k}}'|}\leqslant\int_{R^3}\frac{e^{-|\tilde{\tilde{k}}'-\frac{p_2+1}{p+1}\tilde{\tilde{k}}|^2}\,d\tilde{\tilde{k}}'}{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\cdot|\tilde{\tilde{k}}'|}.$$

We can consider $p_2\leqslant p_1$. Otherwise we replace $\tilde{\tilde{k}}'$ by $\tilde{\tilde{k}}-\tilde{\tilde{k}}'$. There can be several cases.

(b$_1$) $|\tilde{\tilde{k}}|=\frac{|\tilde{k}|\sqrt{p+1}}{\sqrt{(p_1+1)(p_2+1)}}\leqslant 10$. Instead of 10 we could take any sufficiently large number.

In this case

$$\int_{R^3}\frac{e^{-|\tilde{\tilde{k}}'-\frac{p_2+1}{p+1}\tilde{\tilde{k}}|^2}\,d\tilde{\tilde{k}}'}{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\cdot|\tilde{\tilde{k}}'|}\leqslant B_{40}$$

and from (25) to (26)

$$I^{(4)}_{p_1,p_2}\leqslant\frac{B_{41}}{\sqrt{p(p_1+1)(p_2+1)}}\leqslant\frac{10\,B_{41}}{\tilde{k}\cdot(p+1)^{3/2}}.$$

In our case $\frac{p+1}{p_1+1}\leqslant 4$ and from $\frac{|\tilde{k}|\sqrt{p+1}}{\sqrt{(p_1+1)(p_2+1)}}\leqslant 10$ it follows that $\frac{|\tilde{k}|}{\sqrt{p_2+1}}\leqslant 20$ and $\frac{1}{\sqrt{p}}\leqslant\frac{20}{|\tilde{k}|}$. Therefore

$$I^{(4)}_{p_1,p_2}\leqslant\frac{B_{42}}{|\tilde{k}|^2}.\tag{27}$$

(b$_2$) $|\tilde{\tilde{k}}|=\frac{|\tilde{k}|\sqrt{p+1}}{\sqrt{(p_1+1)(p_2+1)}}\geqslant 10$, $|\tilde{\tilde{k}}|\cdot\frac{p_2+1}{p+1}\leqslant 5$.

In this case

$$\int_{R^3} \frac{e^{-|\tilde{\tilde{k}}'-\frac{p_2+1}{p+1}\tilde{\tilde{k}}|^2} d\tilde{\tilde{k}}'}{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\cdot|\tilde{\tilde{k}}'|} \leqslant \frac{B_{42}}{|\tilde{\tilde{k}}|}$$

and since $\frac{1}{p+1} \leqslant \frac{5}{|\tilde{\tilde{k}}|\cdot(p_2+1)}$, $\frac{1}{p_1+1} \leqslant \frac{2}{p+1} \leqslant \frac{10}{|\tilde{\tilde{k}}|(p_2+1)}$ we can write

$$I^{(4)}_{p_1,p_2} \leqslant \frac{B_{39}\cdot B_{42}}{\sqrt{(p+1)(p_1+1)(p_2+1)}\cdot|\tilde{\tilde{k}}|} \leqslant \frac{B_{43}}{|\tilde{\tilde{k}}|^2\cdot(p_2+1)^{3/2}} \leqslant \frac{B_{43}}{|\tilde{\tilde{k}}|^2}.$$

(b$_3$) $|\tilde{\tilde{k}}|\,\frac{p_2+1}{p+1} \geqslant 5$ and therefore $|\tilde{\tilde{k}}| \geqslant 10$.

In this case the main contribution to the integral comes from $\tilde{\tilde{k}}' \sim \frac{p_2+1}{p+1}\tilde{\tilde{k}}$ and therefore

$$\int_{R^3} \frac{e^{-|\tilde{\tilde{k}}'-\frac{p_2+1}{p+1}\tilde{\tilde{k}}|^2} d\tilde{\tilde{k}}'}{|\tilde{\tilde{k}}-\tilde{\tilde{k}}'|\cdot|\tilde{\tilde{k}}'|}$$
$$\leqslant \frac{B_{44}\ (p+1)}{|\tilde{\tilde{k}}|\cdot|\tilde{\tilde{k}}|\cdot(p_2+1)\sqrt{(p+1)(p_1+1)(p_2+1)}} \leqslant \frac{B_{45}}{|\tilde{\tilde{k}}|^2}.$$

Finally we can write

$$I^{(4)}_{p_1,p_2} \leqslant \frac{B_{46}}{|\tilde{\tilde{k}}|^2} = \frac{B_{46}(p_1+1)\,(p_2+1)}{|\tilde{k}|^2\cdot(p+1)}$$

and

$$I_{p_1,p_2} \leqslant \frac{B_{47}\cdot(p_1+1)(p_2+1)}{|\tilde{k}|^2(p+1)}.$$

Thus (see (23))

$$|g_p(\tilde{k},s)| \leqslant e^{-\frac{|\tilde{k}|^2}{p+1}}\cdot\frac{1}{|\tilde{k}|}\cdot\sum_{\substack{p_1,p_2\geqslant 0\\ p_1+p_2=p-1}} C_{p_1}\cdot C_{p_2}\cdot\frac{(p_1+1)(p_2+1)}{(p+1)}.$$

This completes the proof of the theorem.

3. THE DISCUSSION OF THE RESULTS

First we show that the constants C_p grow not faster then exponentially. Denote $C'_p = C_p(p+1)$. The numbers C'_p satisfy the relation

$$C'_p = B \sum_{\substack{p_1, p_2 \geqslant 0 \\ p_1+p_2=p-1}} C'_{p_1} C'_{p_2}.$$

Using the induction, let us prove that $C'_p \leqslant B'(\widetilde{B})^p \cdot \frac{1}{(p+1)^{3/2}}$ for any $p \geqslant 0$ and some constant $\widetilde{B}$. For $p=0$ we can choose $\widetilde{B}$. Assuming that the last inequality is valid for all $q < p$ we can write

$$C'_p \leqslant B \cdot (\widetilde{B})^{p-1} \sum_{\substack{p_1, p_2 \geqslant 0 \\ p_1+p_2=p-1}} \frac{1}{(p_1+1)^{3/2}(p_2+1)^{3/2}} \leqslant B \cdot B_1 \cdot (\widetilde{B})^{p-1} \cdot \frac{1}{(p+1)^{3/2}}.$$

Take $\widetilde{B} = B \cdot B_1$. This gives the result.

Now we see that the series (7) converges if $\lambda = A \cdot t^{\epsilon/2} < (\widetilde{B})^{-1}$.

In many estimates done for the NSS system, people assumed that solutions $v(k,t)$ at infinity in k satisfy the inequality $|v(k,t)| \leqslant e^{-f(t)|k|}$ which is different from the diffusion-like asymptotics. If this asymptotics represents the true decay of solutions, it is an interesting question how does it appear. The series (7) sheds some light on this question. We can write

$$|v(k,t)| \leqslant \frac{\text{Const}}{|k|^{\alpha}} \sum_{p \geqslant 0} \widetilde{B}^p \cdot \lambda^p \cdot \frac{1}{(p+1)^{3/2}} e^{-\frac{t|k|^2}{p+1}}.$$

The usual asymptotical method shows that the largest term in this last sum is when $\frac{t|k|^2}{(p_{\max}+1)^2} = -\ln(b_2\lambda)$, i.e. $p_{\max} = \frac{\sqrt{t} \cdot |k|}{\sqrt{-\ln(b_2\lambda)}}$ and the whole sum behaves as $e^{2p_{\max}\ln(b_2\lambda)} = e^{-2\sqrt{t}\sqrt{-\ln(b_2\lambda)} \cdot |k|}$. This is the asymptotics which was mentioned above. It also shows that in the domain of convergence of the series the enstrophy of the solution is finite for $t > 0$.

This type of decay of solutions in various situations was obtained earlier in the works.[2,6]

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